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NATIONAL BUREAU OF STANDARDS REPORT

1759

ON THE CONVERGENCE OF CYCLIC LINEAR ITERATIONS FOR SYMMETRIC AND NEARLY SYMMETRIC MATRICES. II

by

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and
University of Basle, Switzerland

NBS

U. S. DEPARTMENT OF COMMERCE
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15. **MISSILE DEVELOPMENT.** Missile Engineering. Missile Dynamics. Missile Intelligence. Missile Instrumentation. Technical Services. Combustion.

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9. In the following all notations of the first note under the above title are assumed as known.

In the case of the cyclic single step iteration a converse of Pizzetti's theorem has been found recently, by E. Reich[1]. Reich's theorem is that if in a real symmetric matrix A all diagonal elements are positive and if the cyclic single step iteration converges for A for any choice of the starting vector, then A is a *definite positive* matrix. In what follows we give an essentially simpler proof of Reich's theorem and generalize it to the case of the general linear cyclic iteration. We prove

III. *Let* A *be an Hermitian matrix and let in the notations of* (3) *each of the matrices* P_μ *be a positive definite Hermitian matrix. Then, if the corresponding cyclic linear iteration defined by* (2) *is convergent for any starting vector* ξ_1, *the matrix* A *is a positive definite matrix.*

As a matter of fact our proof of (3) gives an essentially more general result: if a regular matrix A is Hermitian but not positive definite and has positive diagonal elements, then open sets of starting vectors in the n-dimensional space exist for which no single step procedure, either periodic or not periodic, is convergent.

10. We prove first the

Lemma. *Let* $A = (a_{\mu\nu})$ *be an Hermitian matrix of order* n $(a_{\mu\nu} = \bar{a}_{\nu\mu})$ *and* P *a sub matrix of* A *corresponding to the first* s $(s < n)$ *rows and columns of* A. *Assume that* P *is a definite positive*

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[1] The Annals of Mathematical Statistics, XX, (1949), pp. 448-451.

matrix. *Let* $(u_1, \dots, u_n)$ *be a vector and the vector* $(v_1, \dots, v_n)$ *be deduced from the vector* (u_ν) *by the equations*

$$(31)\qquad \begin{aligned} \sum_{\beta=1}^{s} a_{\alpha\beta} v_\beta &= - \sum_{\gamma=s+1}^{n} a_{\alpha\gamma} u_\gamma \qquad && (\alpha = 1, \dots, s) \\ v_\gamma &= u_\gamma && (\gamma = s+1, \dots, n) . \end{aligned}$$

Then, if we form the Hermitian forms corresponding to these vectors

$$(32)\qquad Q_u = \Sigma\, a_{\mu\nu} \bar{u}_\mu u_\nu , \quad Q_v = \Sigma\, a_{\mu\nu} \bar{v}_\mu v_\nu$$

we have

$$(33)\qquad Q_v \le Q_u .$$

To prove this lemma we put

$$(34)\qquad h_\alpha = v_\alpha - u_\alpha \qquad (\alpha = 1, \dots, s) .$$

Then we have from the first of the equation (31)

$$(35)\qquad \sum_{\nu=1}^{n} a_{\alpha\nu} u_\nu = - \sum_{\beta=1}^{s} a_{\alpha\beta} h_\beta \qquad (\alpha = 1, \dots, s) .$$

We have then obviously for the difference $Q_v - Q_u$:

$$(36)\qquad Q_v - Q_u = \sum_{\alpha\beta} a_{\alpha\beta} \bar{h}_\alpha h_\beta + \sum_{\alpha} \bar{h}_\alpha \sum_{\nu=1}^{n} a_{\alpha\nu} u_\nu + \sum_{\alpha} h_\alpha \sum_{\mu=1}^{n} a_{\mu\alpha} \bar{u}_\mu .$$

Here we introduce the value (35) of the second sum in the second right hand term; as to the second sum in the third right hand term, we have in using (35)

$$(37)\quad \sum_{\mu=1}^{n} a_{\mu\nu}\bar{u}_{\mu} = \overline{\sum_{\mu=1}^{n} \bar{a}_{\mu\alpha}u_{\mu}} = \overline{\sum_{\mu=1}^{n} a_{\alpha\mu}u_{\mu}} = -\overline{\sum_{\beta=1}^{s} a_{\alpha\beta}h_{\beta}}$$

$$= -\sum_{\beta=1}^{s} \bar{a}_{\alpha\beta}\bar{h}_{\beta} = -\sum_{\beta=1}^{s} a_{\beta\alpha}h_{\beta}$$

(36) becomes now

$$(38)\quad Q_v - Q_u = \sum_{\alpha,\beta} a_{\alpha\beta}\bar{h}_{\alpha}h_{\beta} - \sum_{\alpha}\bar{h}_{\alpha}\sum_{\beta} a_{\alpha\beta}h_{\beta} - \sum_{\alpha} h_{\alpha}\sum_{\beta} a_{\beta\alpha}\bar{h}_{\beta}$$

and we obtain finally

$$(39)\qquad Q_v - Q_u = -\sum_{\alpha,\beta} a_{\alpha\beta}\bar{h}_{\alpha}h_{\beta} \leqq 0$$

which proves our lemma.

11. This lemma can be immediately generalized in applying to the rows and columns of A a cogradient permutation; then we see that if the indices α and β run independently through a group of s different indices among $(1, \dots, n)$ and γ through the complementary set of indices and the vector $(v_1, \dots, v_n)$ is deduced from the vector $(u_1, \dots, u_n)$ by the equations

$$(40)\qquad \begin{aligned} \sum_{\beta} \alpha_{\alpha\beta}u_{\beta} &= -\sum_{\gamma} a_{\alpha\gamma}u_{\gamma}\,, \\ v_{\gamma} &= u_{\gamma}\,, \end{aligned}$$

we have still the inequality (33) if the principal minor $P = (a_{\alpha\beta})$ of A is positive definite.

The proof of the theorem III is now immediate. In applying successively the operations indicated by the equations (2) we pass obviously from the vector $\xi^{(k)}$ to the vector $\xi^{(k+1)}$ through some intermediate vectors obtained one from another by a sequence of equations analogous to (40); therefore the numbers

$$(41) \qquad Q^{(k)} = \Sigma a_{\mu\nu} \bar{x}_{\mu}^{(k)} x_{\nu}^{(k)}$$

form a *monotonically decreasing* sequence of real numbers. Take now all components y_{ν} of the vector η as 0 and start with such a vector $\xi = \xi^{(o)}$ for which

$$\Sigma a_{\mu\nu} \bar{x}_{\mu}^{(o)} x_{\nu}^{(o)}$$

is *negative*. Then the decreasing sequence $Q^{(k)}$ certainly does not tend to zero and the sequence of the vectors $\xi^{(k)}$ is not convergent since it otherwise would have to converge to the null-vector. This proves the theorem III.

Obviously the theorem III remains also true if the constant vector η is given from the beginning, since it can be brought into the origin by a translation, while the cyclic linear iteration is a covariant process with respect to the translation.

12. Suppose now that the regular matrix A is Hermitian and all its diagonal elements are positive. Then, if the Hermitian form corresponding to A is not positive the argument used in the proof of III can be applied to any sequence of single step operations as described in our lemma for s = 1. We see that in this case there always exist open sets of starting vectors for which no sequence of single step operations is convergent.

In this case even the use of "under or overrelaxation" cannot change the situation. The use of incomplete relaxation with a factor q ($0 < q \leq 2$) corresponds in the notation of our lemma to the use of the vector $(w_1, \ldots, w_n)$ obtained by the formulae

(42) $$w_\nu = (1-q)\, u_\nu + q\, v_\nu$$

or, what is the same,

(43) $$w_\alpha = u_\alpha + q\, h_\alpha\,, \qquad w_\gamma = u_\gamma\,.$$

In this case the formulae (38) and (39) are replaced by

(44) $$\sum_{\mu,\nu} a_{\mu\nu}\, \bar{w}_\mu\, w_\nu - \sum_{\mu,\nu} a_{\mu\nu}\, \bar{u}_\mu\, u_\nu = (q^2 - 2q) \sum_{\alpha,\beta} a_{\alpha\beta}\, \bar{h}_\alpha\, h_\beta \leq 0$$

and our argument remains valid, even if q is varied from one step to another remaining of course in the interval (0,2).

June 30, 1952

THE NATIONAL BUREAU OF STANDARDS

Functions and Activities

The National Bureau of Standards is the principal agency of the Federal Government for fundamental and applied research in physics, mathematics, chemistry, and engineering. Its activities range from the determination of physical constants and properties of materials, the development and maintenance of the national standards of measurement in the physical sciences, and the development of methods and instruments of measurement, to the development of special devices for the military and civilian agencies of the Government. The work includes basic and applied research, development, engineering, instrumentation, testing, evaluation, calibration services, and various scientific and technical advisory services. A major portion of the NBS work is performed for other government agencies, particularly the Department of Defense and the Atomic Energy Commission. The functions of the National Bureau of Standards are set forth in the Act of Congress, March 3, 1901, as amended by Congress in Public Law 619, 1950. The scope of activities is suggested in the listing of divisions and sections on the inside of the front cover.

Reports and Publications

The results of the Bureau's work take the form of either actual equipment and devices or published papers and reports. Reports are issued to the sponsoring agency of a particular project or program. Published papers appear either in the Bureau's own series of publications or in the journals of professional and scientific societies. The Bureau itself publishes three monthly periodicals, available from the Government Printing Office: the Journal of Research, which presents complete papers reporting technical investigations; the Technical News Bulletin, which presents summary and preliminary reports on work in progress; and Basic Radio Propagation Predictions, which provides data for determining the best frequencies to use for radio communications throughout the world. There are also five series of nonperiodical publications: the Applied Mathematics Series, Circulars, Handbooks, Building Materials and Structures Reports, and Miscellaneous Publications.

Information on the Bureau's publications can be found in NBS Circular 460, Publications of the National Bureau of Standards ($1.00). Information on calibration services and fees can be found in NBS Circular 483, Testing by the National Bureau of Standards (25 cents). Both are available from the Government Printing Office. Inquiries regarding the Bureau's reports and publications should be addressed to the Office of Scientific Publications, National Bureau of Standards, Washington 25, D. C.

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